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A Comparison of Equations for Powder Compression

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Several equations representing the relation between the pressure and the apparent volume of a powdered mass under compression, as proposed by various authors, sometimes have essentially the same form as one other, and sometimes have contradictory characters. Various equations can be modified so as to be a function of the porosity and thus compared with each other. As a result, they can be classified into two types by means of the gradient of the porosity with the pressure ($-\mathrm{d}n/\mathrm{d}P$, n=porosity). One is $-\mathrm{d}n/\mathrm{d}P=kn^x$ (Athy: $x=1$, Kawakita: $x=2$), and the other is $-\mathrm{d}n/\mathrm{d}P=k(1-n)^y/P^z$ (Smith: $y=0$, $z=2/3$, Nutting: $y=1$, $z\doteqdot 1$, Balshin: $y=2$, $z\sim 1$, Terzaghi: $y=2$, $z\sim 1$, Jones: $y=3$, $z=1$), where k is a constant and x, y and z are parameters. In Kawakita's equation the rate of the decrease of the porosity is considerably larger in an early period of compaction than in Athy's equation. The second type of equation indicates that a major part of the porosity disappears at lower pressures.

The compacting behavior of a powdered mass was previously analyzed by Kawakita,[1-2] one of the present authors. He proposed an empirical Eq. 1:

$$C=\frac{V_0-V}{V_0}=\frac{abP}{1+bP} \tag{1}$$

where C= the relative reduction of the volume by compression; V_0= the initial apparent volume; V= the volume under the applied pressure P, and a and b= constants.

The constant a corresponds to the limiting value of the relative reduction of the volume by compression: it should be equal to the initial porosity of a powdered mass, as expressed by Eq. 2:

$$a=C_\infty=\frac{V_0-V_\infty}{V_0} \tag{2}$$

where V_∞ is the net volume of the powder. The b constant should be equal to the reciprocal of the pressure when the value, C, reaches one-half of the limiting value ($C=C_\infty/2$).

Several equations of other types have also been proposed for the powder compression by various

1) K. Kawakita, *Science, Japan*, **26**, 149 (1956).
2) K. Kawakita, *Ann. Report Takamine Lab. (Sankyo Co., Tokyo)*, **8**, 83 (1956).
3) K. Kawakita, ibid., **8**, 87 (1956).
4) K. Kawakita, ibid., **8**, 92 (1956).
5) K. Kawakita, ibid., **10**, 93 (1958).
6) K. Kawakita, *J. Japan Soc. Powder Metallurgy*, **10**, 71 (1963).
7) K. Kawakita, *J. Soc. Materials Science, Japan*, **13**, 421 (1964).
8) R. E. Collins, *Transactions American Geophysical Union*, **45**, 161 (1964).
9) K. Kawakita and Y. Tsutsumi, *Japan. J. Appl. Phys.*, **4**, 56 (1965).

TABLE I. POWDER COMPRESSION EQUATIONS

Author	Equation	Remarks
Balshin	$\ln P = -L \cdot V_r + C$	P=applied pressure, V_r=specific volume, L, C=constants
Smith	$\rho - \rho_0 = C_f \cdot P^{1/3}$	ρ=apparent density, ρ_0=initial apparent density, C_f=compressibility factor
Murray	$\ln\left(\frac{1}{1-D_E}\right) = \frac{\sqrt{2\gamma\, n^{1/3}}}{\sigma_0}\left(\frac{D_E}{1-D_E}\right)^{1/3}\frac{4\pi^{1/3}}{3} + \frac{P}{\sqrt{2}\sigma_0}$	D_E=relative density at pressure P, γ=surface tension, n=number of pores in unit volume, σ_0=yield strength at compressing temperature
Ballhausen	$\ln\left(\frac{D}{1-D}\right) \propto P$	D=relative density at pressure P
Konopicky	$\ln\left(\frac{1}{1-D}\right) = kP + \ln\left(\frac{1}{1-D_0}\right)$	D=relative density at pressure P, D_0=relative density at zero pressure, k=constant
Jones	$\frac{dP}{dW} = -kPW$	P=applied pressure, W=relative apparent volume, k=constant
Burr	$D = f_1(P) + f_2(P) + f_3(P) + A$	D=relative apparent density, f_1, f_2, f_3=functions of pressure P, A=constant
Athy	$n = n_0 \cdot \exp(-bx)$	n=porosity, n_0=average porosity of surface clays, b=constant, x=depth of burial
Nutting	$\varepsilon = \psi^{-1} t^{\kappa} f^{\beta}$	ε=volumetric strain, f=compressive stress, t=time, ψ, β, κ=constants
Tanimoto	$\delta = \frac{1-\mu}{E}\sigma + \frac{\sigma\varepsilon_m + A\varepsilon_0}{\sigma + A}$	δ=change of volume, E=elastic factor, σ=compressive stress, ε_m=initial void, ε_0=strain at initial state, μ, A=constants
Terzaghi	$\varepsilon = -\alpha \ln(P+P_c) - \beta(P+P_c) - \gamma P + C$	ε=ratio of porosity, P=applied pressure, $\alpha, \beta, \gamma, P_c, C$=constants
Cooper	$\overset{*}{V} = a_1 \cdot \exp(-k_1/P) + a_2 \cdot \exp(-k_2/P)$	$\overset{*}{V}$=fractional compaction, P=applied pressure, a_1, a_2=fractional coefficients, k_1, k_2=constants
Kawakita	$C = \frac{abP}{1+bP}$	P=applied pressure, C=relative reduction of volume, a, b=constants

Equations from Balshin's to Konopicky's are from Ref. 11, Jones' and Burr's are from Ref. 12, Athy's, Nutting's, Tanimoto's, Terzaghi's and Cooper's are from Refs. 13, 14, 15, 16 and 18 respectively.

investigators, and some of them have been used in practice, though only in a few cases. This paper will deal with our attempt to compare these equations.

Powder Compression Equations.—So far as we are aware, the following equations have been reported (Table I[11-18]).

In order to make a comparison of these equations, each of them should be rewritten by the same notations (Table II).

Burr's equation was not evaluated since no representation was given for the pressure functions.

10) K. Kawakita, Y. Tsutsumi, C. Ikeda and H. Yagi, *Oyo Buturi (Japan)*, **35**, 360 (1965).
11) R. W. Heckel, *Trans. AIME Met.*, **221**, 1001 (1961).
12) M. J. Donachie and M. F. Burr, *J. Metals*, **15**, 849 (1963).
13) L. F. Athy, *Bull. Am. Assoc. Petrol. Geologists*, **14**, 1 (1949).
14) G. W. Scott Blair and F. M. Valda Coppen, *Nature*, **146**, 840 (1940).
15) K. Tanimoto, *Transactions Japan Soc. Civil Engineers*, **43**, 53 (1957); T. Wakabayashi, *J. Japan Soc. Powder Metallurgy*, **10**, 83 (1963).
16) K. Terzaghi, "Erdbaumechanik," Franz Deuticke Publ. Co., Leipzig and Wien (1925), p. 91.
17) B. S. Neumann, "Flow Properties of Disperse Systems," North-Holland Publ. Co., Amsterdam (1953), p. 397.
18) A. R. Cooper and L. E. Eaton, *J. Am. Ceramic Soc.*, **45**, 97 (1962).

As for Nutting's equation, which includes a time-dependent factor, the measuring time was taken as a constant. In Athy's equation we substituted the depth of burial, x, with a pressure term, P.

Murray's, Konopicky's and Tanimoto's Equations.—Murray's equation is rather complicated:

$$\ln\left(\frac{V}{V-V_\infty}\right) = c_4\left(\frac{V_\infty}{V-V_\infty}\right)^{1/3} + c_5 P$$

However, if the first term of the right side is the constant (k), we find:

$$\frac{V-V_\infty}{V} = k \cdot e^{-c_5 P}$$

which is the same in form as Athy's equation. It is possible, therefore, to say that Murray's equation is a sort of modification of Athy's.

The equations of Balshin, Smith, Murray, Ballhausen and Konopicky were evaluated in the light of experimental data provided by Heckel.[11] He concluded that Konopicky's equation agreed best with the data. We have previously stated[9] that Konopicky's equation could be regarded as the same as that of Athy.

Tanimoto's equation may be rewritten as follows:

$$\frac{V_0-V}{V_0} = \frac{\beta P + \gamma P^2}{1+\alpha P}$$

TABLE II. VARIOUS EQUATIONS REPRESENTED BY THE SAME NOTATIONS

Balshin	$\ln P=-c_1\dfrac{V}{V_\infty}+c_2$
Smith	$\dfrac{1}{V}-\dfrac{1}{V_0}=c_3\cdot P^{1/3}$
Murray	$\ln\left(\dfrac{V}{V-V_\infty}\right)=c_4\left(\dfrac{V_\infty}{V-V_\infty}\right)^{1/3}+c_5P$
Ballhausen	$\ln\left(\dfrac{V_\infty}{V-V_\infty}\right)=c_6P+\ln c_7$
Konopicky	$\ln\left(\dfrac{V}{V-V_\infty}\right)=c_8P+\ln\left(\dfrac{V_0}{V_0-V_\infty}\right)$
Jones	$\ln P=-c_9\left(\dfrac{V}{V_\infty}\right)^2+c_{10}$
Athy	$\dfrac{V-V_\infty}{V}=\dfrac{V_0-V_\infty}{V_0}\cdot e^{-c_{11}P}$
Nutting	$\ln\left(\dfrac{V_0}{V}\right)=c_{12}P^{c_{13}}$
Tanimoto	$\dfrac{V_0-V}{V_0}=\dfrac{c_{14}P}{V_0}+\dfrac{c_{16}P}{P+c_{15}}$
Terzaghi	$\dfrac{V-V_\infty}{V_\infty}=-c_{17}\ln(P+c_{18})-c_{19}(P+c_{18})-c_{20}P+c_{21}$
Cooper	$\dfrac{V_0-V}{V_0-V_\infty}=c_{22}\cdot e^{-c_{23}/P}+c_{24}\cdot{}^{-c_{25}/P}$
Kawakita	$\dfrac{V_0-V}{V_0}=\dfrac{c_{25}c_{26}P}{1+c_{26}P}$

V_0=initial apparent volume of powder
V=volume of powder under the applied pressure P
V_∞=net volume of powder
c_1—c_{26} are the constants.

where α, β and γ are the constants. This equation contains the self-contradiction that the volume of powder, V, should approach $-\infty$ at the region of large pressure as a result of the γP^2 term. Our experimental data[10] of the tablet compression of many kinds of powder were applied to this equation, and the coefficients, α, β and γ, were calculated by the method of least squares. As a result, the value of γ was smaller than one several thousandth of the value of either α or β on every occasion. Practically, then, the value of γ can be regarded as zero. If the constant γ is zero, Tanimoto's equation should be the same in form as Kawakita's equation.

A Comparison of Equations Expressed in Terms of Porosity.—Excepting the equations of Murray, Konopicky and Tanimoto, each equation shown in Table II was modified so as to include the same porosity term as a scale.

The apparent compacting volume, V, can be derived from each equation; this volume is used to express the porosity term, $(V-V_\infty)/V$. The results obtained are given in Table III. In Cooper's equation, we considered only the first term in the right side.

As is demonstrated in Table III, Balshin's and Jones' equations can only be applied in some limited ranges of pressure, because the degree of porosity must assume an unreasonable value, 100%, if the compacting pressure is either very low or very high. Similarly, Smith's, Nutting's, Terzaghi's and Cooper's equations cannot be applied to the higher pressure range. Thus, concerning the change in porosity, it is shown that only Ballhausen's, Athy's and Kawakita's equations are applicable over the entire range of pressure.

Porosity Gradient with Applied Pressure.—With regard to the rate of porosity change as a function of the pressure, some differences exist among the equations in question, as is shown below.

TABLE III. COMPARISON OF POROSITIES DERIVED FROM VARIOUS EQUATIONS

Author	Porosity (n=)	$P\to0$	$P\to\infty$	Necessary condition
Balshin	$1-\dfrac{c_1}{c_2-\ln P}$	1	1	$c_1\neq0$
Smith	$n_0-c_3V_\infty P^{1/3}$	n_0	$-\infty$ ($c_3>0$) $+\infty$ ($c_3<0$)	$c_3\neq0$
Ballhausen	$\dfrac{1}{1+c_7\cdot\exp(c_6P)}$	$\dfrac{1}{1+c_7}$	0	$c_6\neq0$ $c_7\neq0$
Jones	$1-\left(\dfrac{c_9}{c_{10}-\ln P}\right)^{1/2}$	1	1	$c_9\neq0$
Athy	$n_0\cdot e^{-c_{11}P}$	n_0	0	$c_{11}\neq0$
Nutting	$1-\left(\dfrac{V_\infty}{V_0}\right)e^{c_{12}P^{c_{13}}}$	n_0	n_0 ($c_{13}<0$) $-\infty$($c_{12}>0, c_{13}>0$) 1 ($c_{12}<0, c_{13}>0$)	$c_{12}\neq0$ $c_{13}\neq0$
Terzaghi	$1-\dfrac{1}{1-c_{17}\ln(P+c_{18})-c'_{19}(P+c_{18})+c'_{21}}$	$1-\dfrac{1}{1-c_{17}\ln c_{18}-c'_{19}c_{18}+c'_{21}}$	1	
Cooper	$1-\dfrac{(V_\infty/V_0)}{1-n_0c_{22}\cdot\exp(-c_{23}/P)}$	n_0	$\dfrac{1-c_{22}}{(1/n_0)-c_{22}}$	$c_{22}\neq0$ $c_{23}\neq0$
Kawakita	$\dfrac{n_0}{1+(V_\infty/V_0)c_{26}P}$	n_0	0	$c_{26}\neq0$

n=porosity, n_0=initial porosity

Balshin : $-\frac{dn}{dP} = a\cdot(1-n)^2P^{-1}$

Smith : $-\frac{dn}{dP} = b\cdot P^{-2/3}$

Ballhausen: $-\frac{dn}{dP} = c\cdot(1-n)\cdot n$

Jones : $-\frac{dn}{dP} = d\cdot(1-n)^3\cdot P^{-1}$

Athy : $-\frac{dn}{dP} = e\cdot n$

Nutting : $-\frac{dn}{dP} = f\cdot(1-n)\cdot P^{c_{13}-1}$

Terzaghi : $-\frac{dn}{dP} = (1-n)^2\cdot\left(\frac{c_{17}}{P+c_{18}}+c_{19}\right)$

Cooper : $-\frac{dn}{dP} = g\cdot\frac{(1-n)}{P^2}\cdot(n_0-n)$

Kawakita : $-\frac{dn}{dP} = h\cdot n^2$

where n is the porosity at the applied pressure, P; n_0, the initial porosity, and a, b, c, d, e, f, g and h are the constants. According to Neumann,[17] the c_{13} value in Nutting's equation is very small, that is, the order of 0.1. Furthermore, Terzaghi's equation becomes identical with Balshin's if the c_{18} and c_{19} constants are very small.

Thus, these equations may be represented by two different types of equations:

(I): $-\frac{dn}{dP} = k\cdot n^x$: Athy ($x=1$), Kawakita ($x=2$)

(II): $-\frac{dn}{dP} = k\cdot\frac{(1-n)^y}{P^z}$: Smith ($y=0$, $z=2/3$), Nutting ($y=1$, $z\doteqdot1$), Balshin ($y=2$, $z=1$), Terzaghi ($y=2$, $z\sim1$), Jones ($y=3$, $z=1$)

where k is a proportional constant. Ballhausen's equation is a multiplication of the types I and II ($x=1$, $y=1$, $z=0$). Cooper's equation is the sum of the type II and the type I×II.

From these equations we note that the rate of the change in porosity due to the pressure is related to the ratio of the void to the volume of powder in Athy's and Kawakita's equations, while it is related to the applied pressure and the ratio of the net volume in the other equations.

Figures 1(A) and (B) illustrate the porosity change as a function of the pressure. In these figures we picked out Athy's ($x=1$) and Kawakita's ($x=2$) equations from the type I, Smith's ($y=0$), Nutting's ($y=1$), Balshin's ($y=2$) and Jones' ($y=3$) equations from the type II, and Ballhausen's from the type I×II.

The initial porosity was taken to be 80 and 40% for A and B respectively, while at 3000 kg./cm² it was taken to be 10% in both cases. Even if these percentages are taken otherwise, the general trend of the curves should be the same.

When we made Fig. 1, we used the value of 0.1 for c_{13} in Nutting's equation. In the case of Balshin's and Jones' equations, we constructed the curves for the pressure higher than 0.001 kg./cm², because in both equations the porosity become 100% when the pressure was zero.

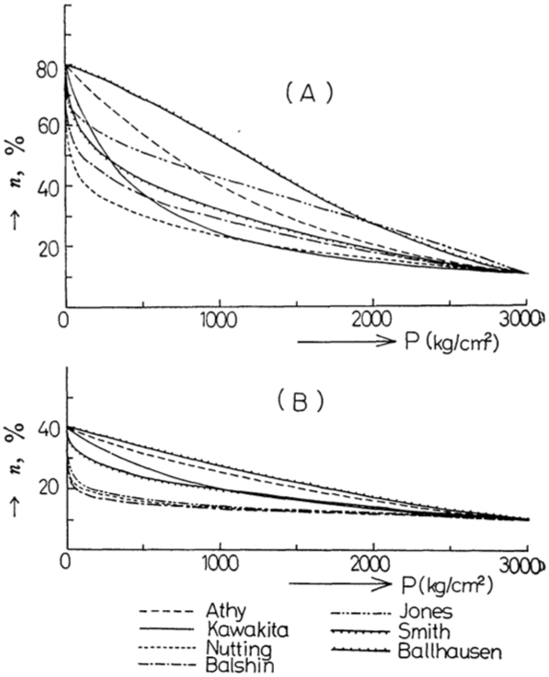

Fig. 1 (A), (B). Comparison of various equations using the term of porosity.

It may, thus, be concluded from Fig. 1 that Ballhausen's equation predicts the most gradual decrease in porosity as the pressure increases, and that the next most gradual decrease is in Athy's equation. Kawakita's equation indicates a considerable decrease in porosity in an early period of compaction. The other equations show that a major portion of the porosity disappears at lower pressures.

According to our experiments on tablet compression,[10] powder of common nonspherical particles changed its porosity largely in an early period of compaction; its compacting behavior was in good agreement with Kawakita's equation. On the other hand, powder of spherical glass or metal particles did not change its porosity appreciably as a result of the pressure; therefore, Athy's equation became applicable.

Finally, the compacting behavior of powder depends remarkably on the experimental conditions and on the physical properties of particles;

therefore, it is difficult to find a single equation which is in exact accord with the experimental data on different kinds of powder and different experimental conditions.

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